Sections 8.1, 10.1 and 10.2:

Inner Product Spaces
&
The Gram-Schmidt Orthonormalization Process

Ideas in this section…

1) The Dot Product and Norm in $\mathbb{R}^n$ and it's properties
2) The Gram-Schmidt Orthonormalization Process in $\mathbb{R}^n$
3) Inner-Product Spaces and Norms
4) The Gram-Schmidt Orthonormalization Process in Inner-Product Spaces

The Dot Product and Norm in $\mathbb{R}^n$ and it's Properties

Def: If $\vec{v} = (a_1, a_2, \dots, a_n)$ and $\vec{w} = (b_1, b_2, \dots, b_n)$ are 2 vectors in $\mathbb{R}^n$, then their dot product is the real number defined by…

$$\vec{v} \cdot \vec{w} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

"Defining" Properties of the Dot Product: $\forall\ \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $k \in \mathbb{R}$,…

P1: $\boldsymbol{\vec{v} \cdot \vec{w} \in \mathbb{R}}$

P2: $\boldsymbol{\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}}$

P3: $\boldsymbol{\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}}$

P4: $\boldsymbol{(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w})}$

P5: $\boldsymbol{\vec{v} \cdot \vec{v} \geq 0}$ and $\boldsymbol{\vec{v} \cdot \vec{v} = 0}$ iff $\boldsymbol{\vec{v} = \vec{0}}$

The Dot Product and Norm in $\mathbb{R}^n$ and it's Properties

Prove property P1 for vectors in $\mathbb{R}$:

P1: $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} \in \mathbb{R}$

The Dot Product and Norm in $\mathbb{R}^n$ and it's Properties

Prove property P2 for vectors in $\mathbb{R}$:

P2: $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} = \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}}$

The Dot Product and Norm in $\mathbb{R}^n$ and it's Properties

Prove property P3 for vectors in $\mathbb{R}$:

P3: $\vec{\boldsymbol{u}} \cdot (\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}}) = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} + \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{w}}$

The Dot Product and Norm in $\mathbb{R}^n$ and it's Properties

Prove property P4 for vectors in $\mathbb{R}$:

P4: $(\boldsymbol{k}\vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{w}} = \boldsymbol{k}(\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}})$

The Dot Product and Norm in $\mathbb{R}^n$ and it's Properties

Prove property P5 for vectors in $\mathbb{R}$:

P5: $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} \geq \mathbf{0}$ and $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} = \mathbf{0}$ iff $\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$

The Dot Product and Norm in $\mathbb{R}^n$ and it's Properties

Other things to know about the dot product:

1) If $\vec{v}$ is a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, it can be shown that its length is $\sqrt{\vec{v} \cdot \vec{v}}$.

So, for any vector $\vec{v} \in \mathbb{R}^n$, we define its norm (or length or magnitude or absolute value or modulus) to be

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

and so $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

Note: We say that a vector $\vec{v} \in \mathbb{R}^n$ is a unit vector if its length is 1. In this case…

$$\|\vec{v}\| = 1 \quad \text{or} \quad \vec{v} \cdot \vec{v} = 1$$

The Dot Product and Norm in $\mathbb{R}^n$ and it's Properties

Other things to know about the dot product:

2) If $\vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and θ is the angle between the vectors, it can be shown that

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$$

So, for any nonzero pair of vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, we define the angle between the vectors θ as

$$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$$

Note: We say nonzero vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ are orthogonal (or perpendicular) if the angle between them is 90°. In this case…

$$\vec{v} \cdot \vec{w} = 0$$

The Dot Product and Norm in $\mathbb{R}^n$ and it's Properties

Other things to know about the dot product:

3) If $\vec{v}$ is a nonzero vector but isn't a unit vector, by normalizing a vector, we mean to find a vector in the same direction as $\vec{v}$ but with length 1.

The normalized version of $\vec{v}$ is $\frac{1}{||\vec{v}||}\vec{v}$

The Dot Product and Norm in $\mathbb{R}^n$ and it's Properties

<u>Ex 1</u>: Let $\vec{u} = (3,1,-2,7)$, $\vec{v} = (-4,2,0,3)$, and $\vec{w} = (2,1,9,2)$

a) Find $\vec{u} \cdot \vec{v}$

b) Find $||\vec{v}||$

c) Find the angle between $\vec{u}$ and $\vec{v}$

The Dot Product and Norm in $\mathbb{R}^n$ and it's Properties

<u>Ex 1</u>: Let $\vec{u} = (3,1,-2,7)$, $\vec{v} = (-4,2,0,3)$, and $\vec{w} = (2,1,9,2)$

d) Are vectors $\vec{v}$ and $\vec{w}$ orthogonal?

e) Normalize vector $\vec{v}$

The Gram-Schmidt Orthonormalization Process in $\mathbb{R}^n$

Def: Let U be a subspace of $\mathbb{R}^n$ and let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ be a basis for U. Then $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is an orthonormal basis for U if…

1) $\vec{w}_i$ is orthogonal to $\vec{w}_j$ for all $i \neq j$
   (that is, $\vec{w}_i \cdot \vec{w}_j = 0$ for all $i \neq j$)

2) Each $\vec{w}_i$ has norm 1
   (that is, $\vec{w}_i \cdot \vec{w}_i = 1$ for all i)

Let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ be linearly independent vectors in $\mathbb{R}^n$ and let $U = \text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$. (So $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a basis for U)

The goal is to construct an orthonormal basis $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ for U using vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$

The Gram-Schmidt Orthonormalization Process in $\mathbb{R}^n$

Given linearly independent vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$

Step 1: Come up with a set of orthogonal vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ by calculating…

$$\vec{v}_1 = \vec{u}_1$$

$$\vec{v}_2 = \vec{u}_2 - proj_{\vec{v}_1}\vec{u}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$\vec{v}_3 = \vec{u}_3 - proj_{\vec{v}_1}\vec{u}_3 - proj_{\vec{v}_2}\vec{u}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$\vdots$$

$$\vec{v}_k = \vec{u}_3 - proj_{\vec{v}_1}\vec{u}_3 - proj_{\vec{v}_2}\vec{u}_3 \ldots - proj_{\vec{v}_{k-1}}\vec{u}_3$$

$$= \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 \ldots - \frac{\vec{u}_3 \cdot \vec{v}_{k-1}}{\|\vec{v}_{k-1}\|^2}\vec{v}_{k-1}$$

The Gram-Schmidt Orthonormalization Process in $\mathbb{R}^n$

Given linearly independent vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$

<u>Step 2</u>: Divide each $\vec{v}_i$ by its length to normalize it. That is, $\vec{w}_i = \frac{1}{\|\vec{v}_i\|}\vec{v}_i$.

Then $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ will be the desired orthonormal basis for $U = span\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$

Talk about why this step doesn't change the orthogonality created in step 1

The Gram-Schmidt Orthonormalization Process in $\mathbb{R}^n$

Ex 2: Use the Gram-Schmidt Orthonormalization procedure to find an orthonormal basis for $\mathbb{R}^2$ starting with the independent set of vectors $\{ (2,-1) ,\ (7,4) \}$

The Gram-Schmidt Orthonormalization Process in $\mathbb{R}^n$

Ex 3: Use the Gram-Schmidt Orthonormalization procedure to find an orthonormal basis for $U = span\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ where $\vec{u}_1 = (1,1,-1,-1)$, $\vec{u}_2 = (3,2,0,1)$, and $\vec{u}_3 = (1,0,1,0)$.

Inner-Product Spaces and Norms

Def: Let V be a vector space. An inner-product is a function that takes any pair of vectors in V and returns a scalar (real number) that satisfies the properties below. If $\vec{v}$ and $\vec{w}$ are vectors in V, their inner-product is denoted by $< \vec{v}, \vec{w} >$.

Defining Properties of an Inner-Product: $\forall\ \vec{u}, \vec{v}, \vec{w} \in V$ and $k \in \mathbb{R}$,…

P1: $< \vec{v}, \vec{w} > \in \mathbb{R}$

P2: $< \vec{v}, \vec{w} > = < \vec{w}, \vec{v} >$

P3: $< \vec{u}, \vec{v} + \vec{w} > = < \vec{u}, \vec{v} > + < \vec{u}, \vec{w} >$

P4: $< \vec{v}, k\vec{w} > = k < \vec{v}, \vec{w} >$

P5: $< \vec{v}, \vec{v} > \geq 0$ and $< \vec{v}, \vec{v} > = 0$ iff $\vec{v} = \vec{0}$

A vector space V with an inner-product defined on it is called an inner-product space.

Inner-Product Spaces and Norms

Def: Let V be an inner-product space with inner-product $<\cdot,\cdot>$.

1) The norm of a vector $\vec{v} \in V$ is defined as

$$\|\vec{v}\| = \sqrt{< \vec{v}, \vec{v} >}$$

2) If $\vec{v}$ and $\vec{w}$ are nonzero vectors in V, the angle between the vectors is defined as

$$\theta = \cos^{-1}\left(\frac{< \vec{v}, \vec{w} >}{\|\vec{v}\|\|\vec{w}\|}\right)$$

3) Nonzero vectors $\vec{v}$ and $\vec{w}$ are orthogonal if

$$< \vec{v}, \vec{w} > = 0$$

4) Vector $\vec{v}$ is a unit vector if

$$\|\vec{v}\|=1 \qquad \text{or} \qquad < \vec{v}, \vec{v} > = 1$$

Inner-Product Spaces and Norms

Ex 4a: Verify that $< f, g > = \int_0^1 f(x)g(x)dx$ is an inner-product on $C[0,1]$.

P1: $< \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} > \in \mathbb{R}$

Inner-Product Spaces and Norms

Ex 4a: Verify that $< f, g > = \int_0^1 f(x)g(x)dx$ is an inner-product on $C[0,1]$.

P2: $< \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} > = < \vec{\boldsymbol{w}}, \vec{\boldsymbol{v}} >$

Inner-Product Spaces and Norms

Ex 4a: Verify that $< f, g > = \int_0^1 f(x)g(x)dx$ is an inner-product on $C[0,1]$.

P3: $< \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} > = < \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} > + < \vec{\boldsymbol{u}}, \vec{\boldsymbol{w}} >$

Inner-Product Spaces and Norms

Ex 4a: Verify that $< f, g > = \int_0^1 f(x)g(x)dx$ is an inner-product on $C[0,1]$.

P4: $\boldsymbol{< \vec{v}, k\vec{w} > = k \ < \vec{v}, \vec{w} >}$

Inner-Product Spaces and Norms

Ex 4a: Verify that $< f, g > = \int_0^1 f(x)g(x)dx$ is an inner-product on $C[0,1]$.

P5: $\boldsymbol{< \vec{v}, \vec{v} > \ \geq 0}$ **and** $\boldsymbol{< \vec{v}, \vec{v} > = 0}$ **iff** $\boldsymbol{\vec{v} = \vec{0}}$

Inner-Product Spaces and Norms

<u>Ex 4b</u>: Using $< f, g > = \int_0^1 f(x)g(x)dx$ as the inner-product on $C[0,1]$,

find 2 nonzero vectors that are orthogonal.

Inner-Product Spaces and Norms

Ex 4c: Using $< f, g > = \int_0^1 f(x)g(x)dx$ as the inner-product on $C[0,1]$,

find a unit vector.

Inner-Product Spaces and Norms

<u>Ex 4d</u>: Using $<f,g> = \int_0^1 f(x)g(x)dx$ as the inner-product on $C[0,1]$,

find the angle between $f(x) = x$ and $g(x) = e^x$.

Inner-Product Spaces and Norms

Ex 5: Show that $< f, g > = \int_0^{1/2} f(x)g(x)dx$ is NOT an inner-product on $C[0,1]$.

P1: $< \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} > \in \mathbb{R}$

P2: $< \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} > = < \vec{\boldsymbol{w}}, \vec{\boldsymbol{v}} >$

P3: $< \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} > = < \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} > + < \vec{\boldsymbol{u}}, \vec{\boldsymbol{w}} >$

P4: $< \vec{\boldsymbol{v}}, \boldsymbol{k}\vec{\boldsymbol{w}} > = \boldsymbol{k} \ < \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} >$

P5: $< \vec{\boldsymbol{v}}, \vec{\boldsymbol{v}} > \ \geq \mathbf{0}$ **and** $< \vec{\boldsymbol{v}}, \vec{\boldsymbol{v}} > = \mathbf{0}$ **iff** $\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$

The Gram-Schmidt Orthonormalization Process in Inner-Product Space V

<u>Def</u>: Let U be a subspace of V and let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ be a basis for U. Then $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is an <u>orthonormal basis</u> for U if…

1) $\vec{w}_i$ is orthogonal to $\vec{w}_j$ for all $i \neq j$
   (that is, $< \vec{w}_i, \vec{w}_j > = 0$ for all $i \neq j$)

2) Each $\vec{w}_i$ has norm 1
   (that is, $< \vec{w}_i, \vec{w}_i > = 1$ for all i)

Let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ be linearly independent vectors in V and let $U = \text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$. (So $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a basis for U)

The goal is to construct an orthonormal basis $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ for U using vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$

The Gram-Schmidt Orthonormalization Process in Inner-Product Space V

Given linearly independent vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$

Step 1: Come up with a set of orthogonal vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ by calculating…

$$\vec{v}_1 = \vec{u}_1$$

$$\vec{v}_2 = \vec{u}_2 - proj_{\vec{v}_1}\vec{u}_2 = \vec{u}_2 - \frac{< \vec{u}_2, \vec{v}_1 >}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$\vec{v}_3 = \vec{u}_3 - proj_{\vec{v}_1}\vec{u}_3 - proj_{\vec{v}_2}\vec{u}_3 = \vec{u}_3 - \frac{<\vec{u}_3,\vec{v}_1>}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{<\vec{u}_3,\vec{v}_2>}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$\vdots$$

$$\vec{v}_k = \vec{u}_3 - proj_{\vec{v}_1}\vec{u}_3 - proj_{\vec{v}_2}\vec{u}_3 \ldots - proj_{\vec{v}_{k-1}}\vec{u}_3$$

$$= \vec{u}_3 - \frac{<\vec{u}_3,\vec{v}_1>}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{<\vec{u}_3,\vec{v}_2>}{\|\vec{v}_2\|^2}\vec{v}_2 \ldots - \frac{<\vec{u}_3,\vec{v}_{k-1}>}{\|\vec{v}_{k-1}\|^2}\vec{v}_{k-1}$$

The Gram-Schmidt Orthonormalization Process in Inner-Product Space V

Given linearly independent vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$

Step 2: Divide each $\vec{v}_i$ by its length to normalize it. That is, $\vec{w}_i = \frac{1}{\|\vec{v}_i\|}\vec{v}_i$.

Then $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ will be the desired orthonormal basis for
$U = span\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$

The Gram-Schmidt Orthonormalization Process in Inner-Product Space V

<u>Ex 6</u>: Use the Gram-Schmidt Orthonormalization procedure to find an orthonormal basis for $U = span\{x, e^x\}$ as a subspace of $C[0,1]$ with inner-product $< f, g > = \int_0^1 f(x)g(x)dx$

What you need to know from the book

Book reading

Section 8.1: pages 407 – top half of 410

Section 10.1: pages 527 – 534

Section 10.2: pages 536-539

Problems you need to know how to do from the book

Section 8.1: page 414 #'s 1, 4

Section 10.1: page 534 #'s 1-3, 5-9, 15-16, 22-23, 26-28, 31

Section 10.2: page 543 #'s 1-5, 10-11, 15-18